Measurement Coding for Communication Limited Control is Optimal Quantisation of the Control Signal

Thomas S Brinsmead[†⋆]

† RSISE, Australian National University,
Canberra ACT 0200, Australia
⋆ email Thomas.Brinsmead@anu.edu.au
Version Number 1.0

1 Introduction

The interdependence of communication and control functions in practical networks is obvious. The two functions are closely and intimately intertwined. Since communication services are required for control it follows that communications constraints will affect the acheivable performance of a controlled system. Conversely, it is possible to interpret communication algorithms as essentially the control of communication channels and infrastructure. Despite their close functional connection, research in these two fields has proceeded relatively independently. Traditional control research, for example, rarely assumes less than perfect communication, and much research in communications proceeds without reference to the control function of the information that is being communicated.

The practical utility of a theoretical understanding of the relationship between control and communications is obvious in the context of designing effective C^3I (Commond, Control, Communications and Intelligence) Systems for organisational management. Under a broad interpretation of a “control function” as “making decisions regarding what actions to undertake on the basis of observation and measurement”, the value of any communicated information is a function only of how it contributes to the improvement of the controlled outcome.

The design (and control) of a communications network is a tradeoff between varying levels of service between nodes, constrained by resources such as available channel capacity and other items of communication infrastructure. The value of information in terms of achievable control performance is a desirable parameter on which to base rational decisions regarding the allocation of such communications resources. One way to determine the value of such control information is to perform optimisation on various criteria under a variety of communication constraint limitations in order to observe how controller performance degrades under such constraints.

Despite the relative lack of research concerned with communication for control, there are a number of exceptions which address the control problem with communications constraints. The problem of stabilising an unstable system under data rate constraints is dealt with for example in [5], [8] and [9]. The conclusion is that there exists a particular data rate threshold above which stabilisation of a linear unstable system is possible, and below which it is impossible. The data rate threshold is a function of time-constant of the fastest unstable pole of the open loop unstable system. The LQG problem with communication constraints is dealt with in [1], [7], and [6]. Others [4] have looked at methods of optimally coding linear system transfer functions under finite data storage limits.

The presentation for the ARO conference will address a problem from a similar perspective: that of a traditional control problem, modified by the addition of communications constraints. It will be certainly a long way from a unifying theory of communications and control, but it is hoped that the insights obtained will contribute towards such a goal. The problem to be addressed here is that of the optimal control of a dynamic system with some constraints on the data rate.

Classical control goals of optimal control will be considered: namely the LQG and $\mathcal{H}_\infty$ cost criteria. Although the communication limited optimal LQG problem has already been solved [1], [7], we present an alternative perspective from which to view the problem which allows the same framework to be employed for solving the optimal $\mathcal{H}_\infty$ problem. This is motivated by a particular observation that the optimal coding of observations y for the purposes of a specified optimal control function is equivalent to the problem of optimal coding of the control input u. Rigourous explanation of this claim will necessitate the introduction of some mathematical notation.

2 Mathematical Preliminaries

- A plant G is modelled as an operator from input space to output space as $G : [u, w] \to [z, y]$ where $w \in \mathcal{W}$ are exogenous signals, $u \in \mathcal{U}$ is a control signal which is able to be manipulated, $z \in \mathcal{Z}$

is an output variable of interest in determining closed loop performance and $y \in \mathcal{Y}$ is available for measurement.

- A finite information space $\mathcal{I}_k$ of index k is a set with a finite number of elements c_i for $1 \leq i \leq f$. Each element c_i in a finite information space may be interpreted as a *codeword.*
- A *codebook* C is a function $C : x \to c_i$ where $x \in \mathcal{X}$ is the *message* and $c_i \in \mathcal{I}_f$ is the *coded message.* In general, a codebook is many-to-one, often mapping an infinite message space to the finite information space.
- A decoder D is a function $D : c_i \to x$ from c_i in a finite information space $\mathcal{I}_f$ to some element x of an output space $\mathcal{X}$.
- A controller $K : y \to u$ maps observations $y \in \mathcal{Y}$ to control inputs $u \in \mathcal{U}$.
- The composition of a plant G and controller K results in a closed loop map $T : w \to z$ from $w \in \mathcal{W}$ to $z \in \mathcal{Z}$.
- A closed loop objective function is a functional $J : T \to \mathbb{R}$ from a closed loop transfer function to a real objective value.
- A discrete time vector sequence $z^\infty \in \mathbb{Z} \times \mathbb{R}^n$ is a sequence of vectors $z_k \in \mathbb{R}^n$ for $0 \leq k \leq \infty$.
- A partial discrete time vector sequence $z^K \in \mathcal{I}_K \times \mathbb{R}^n$ up to time K is a sequence of vectors $z_k \in \mathbb{R}^n$ for $0 \leq k \leq K$.
- A time limited observation operator $O_K : \mathbb{Z} \times \mathbb{R}^n \to \mathcal{I}_K \times \mathbb{R}^n$ is defined in the obvious way by $(O_K z)_k = z_k$ for $k \leq K$.

Proposition 2.1 *Assume that we are given a system $G : [u, w] \to [z, y]$. For simplicitity, we take the input and output spaces $\mathcal{U} = \mathbb{Z} \times \mathbb{R}^m$, $\mathcal{W} = \mathbb{Z} \times \mathbb{R}^r$, $\mathcal{Z} = \mathbb{Z} \times \mathbb{R}^q$, $\mathcal{Y} = \mathbb{Z} \times \mathbb{R}^p$ to be discrete time sequences of vector signals. We are also given an objective function J and a communication constraint in terms of bit rate per sample limit $b \in \mathbb{Z}$.*

We are required to find a set of observation point code books $C_{yk} : \mathcal{I}_k \times \mathbb{R}^p \to \mathcal{I}_{2^b}$ which defines an observation sequence code book $C_y : \mathbb{Z}\times\mathbb{R}^p \to \mathbb{Z}\times\mathcal{I}_{2^b}$ in the obvious way by $(C_y y)_k = C_{yk} O_k y$ and a codebook controller $\bar{K} : \mathbb{Z}\times\mathcal{I}_{2^b} \to \mathcal{U}$, such the effective controller $K = \bar{K} C_y$ optimises the objective function J.

This is equivalent to the problem of finding a controller $\hat{K} : \mathcal{Y} \to \mathcal{U}$, a set of control point code books $C_{uk} : \mathcal{I}_k \times \mathbb{R}^m \to \mathcal{I}_{2^b}$ which defines a control sequence code book $C_u : \mathbb{Z} \times \mathbb{R}^m \to \mathbb{Z} \times \mathcal{I}_{2^b}$ in the obvious way by $(C_u u)_k = C_{uk} O_k u$ and a sequence of decoders $D_{uk} : \mathcal{I}_{2^b} \to \mathbb{R}^m$ which defines a sequence decoder $D_u : \mathbb{Z} \times \mathcal{I}_{2^b} \to \mathbb{Z} \times \mathbb{R}^m$ in the obvious way by $(D_u u)_k = D_{uk} u_k$, such that the effective controller $K = D_u C_u \hat{K}$ optimises the objective function J.

Proof: It is fairly easy to show that for every $\bar{K}$ and C_y there exists a K, D_u and C_u such that $D_u C_u \hat{K} = \bar{K} C_y$. The converse also holds. ∎

Although the proposition statement and proof are technically trivial, the insight that is provided is not. Rather than designing a optimal combination of coding scheme for the measured output y and controller for the coded measurements, a conceptually simpler, but mathematically equivalent problem is to design an optimal combination of controller for observed measurements and encoder for controlled inputs u. Furthermore, if the objective function J obeys a particularly simple convexity property, it follows that the optimal controller for the observed measurements can be designed independently of the code book. This is formalised in the following proposition.

Proposition 2.2 *Assume that the plant G admits an internal state representation, that is, it can be factorised as $G = FH$, where $H : \mathcal{U} \times \mathcal{W} \to \mathcal{X} = \mathbb{Z} \times \mathbb{R}^n$ and $F : \mathcal{X} \to \mathcal{Y}$,. Assume also that the objective function can be expressed as $J = \sum_{k=1}^{\infty} L(x_k, u_k)$ with an associated value function $V_k(x_k) = \mathrm{opt}_u \sum_{i=k}^{\infty} L(x_i, u_i)$. If this value function is convex with respect to u, that is, provided that*

$$\begin{aligned} &\lambda[L(x_k, u_k(a)) + V_{k+1}(x_{k+1})] \\ &+(1-\lambda)[L(x_k, u_k(b)) + V_{k+1}(x_{k+1})] \\ \geq\; & L(x_k, \lambda u_k(a) + (1-\lambda)u_k(b)) + V_{k+1}(x_{k+1}), \end{aligned}$$

then the problem described in Proposition 2.1 may be solved as a combination of the unconstrained optimal control problem to find $\hat{K} : \mathcal{Y} \to \mathcal{U}$ and the optimal coding problem to find $D_u C_u : \mathcal{U} \to \mathcal{U}$ which minimises the distortion in u.

The optimal communication limited control problem thus reduces, in these special convex cases, to an independent optimal controller design and optimal quantisation problem of the sort dealt with in [3]. Applying the above proposition to the classical LQG problem we recover the solution in [7]. Because the value function of the classical $\mathcal{H}_\infty$ problem also enjoys the convexity property condition required in Proposition 2.2, it is also possible to use this fact to solve the communication limited $\mathcal{H}_\infty$ problem.

At present only numerical solutions to the general optimal quantisation problem exist [3], however, these are sufficient to obtain upper and lower bounds on the increased cost to optimal control if there are added bandwidth constraints on communication.

3 Conclusion

The main contribution is the observation that the optimal observation coding for optimal control problem may be recast as an optimal control plus optimal control coding problem to solve a classical optimal control problem with communication constraints. If the value function for the classical optimal control problem enjoys a particular convexity property with respect to the control signal, then the combined optimal control and coding problems may be solved independently. This suggests that when communication limitations are an in issue, the design of an appropriate set of allowable discrete controls on which to base a hybrid controller analysis and design as in [2] can be viewed from a perspective as an optimal coding problem.

In this sense, we make a small contribution to an information theoretic interpretation of the control problem, but there are some other obvious questions for future research. It is well known that the existence of noise can imply certain limitations in terms of rate of information transfer over a communication channel. On the other hand, measurement noise is also explicitly dealt with in the classical LQG and $\mathcal{H}_\infty$ formulations. It remains to be determined whether there is a simple link between the degradation in the controlled performance caused by *communication bandwidth limits* induced by noise and the degradation directly attributable to the noise itself in the classical formulations. Other issues to be investigated relate to the information content in a bandwidth limited continuous time rather than a discrete time signal and to show whether the Shannon channel capacity and Nyquist sampling theorems can be used to lend insight into the continuous time case. Finally there is also the issue of the extent to which performance might be improved by conditioning control on the decoding of a partially observed codeword sequence, instead of waiting until the entire codeword has been received before performing the decoding.

References

[1] V. S. Borkar and S. K. Mitter. LQG control with commmunication constraints. In A. Paulraj, V. Roychowdhury, and C. D. Schaper, editors, *Communications, Computation, Control and Signal Processing: A Tribute to Thomas Kailath*, pages 365–373. Kluwer, NY, 1997.

[2] J. M. Davoren and T. Moor. Logic-based design and synthesis of controllers for hybrid systems. Technical report, Dept. Systems Engineering, RSISE, ANU, July 2000.

[3] A. Gersho and R. M. Gray. *Vector Quantization and Signal Compression.* Kluwer Academic Publishers, Boston, 1992.

[4] A. G. Madievski, B. D. O. Anderson, and M. Gevers. Optimum realizations of sampled-data control for FWL sensitivity minimization. *Automatica*, 31(3):367–379, March 1995.

[5] G. N. Nair and R. J. Evans. Asymptotic stabilizability in communication limited control. Technical report, Dept. of Electrical and Electronic Eng., Univ. Melbourne 3010, Australia, April 2000.

[6] K. Shoarinejad, I. Kanallakopoulos, J. D. Wolfe, and J. L. Speyer. A two-station decentralised lqg problem with noisy communication. *Proc. 38th IEEE Conf on Decision and Control*, pages 4929–4932, December 1999.

[7] S. Tatikonda, A. Sahai, and S. Mitter. Control of LQG systems under communication constraints. *Proc.* 37th *IEEE Conf. on Decision and Control*, pages 1165–1170, December 1998.

[8] W. S. Wong and R. W. Brockett. Systems with finite communication bandwith constraints– I: State estimation problems. *IEEE Trans. Auto. Control*, 42(9):1294–1299, September 1997.

[9] W. S. Wong and R. W. Brockett. Systems with finite communication bandwith constraints– II: Stabilization with limited information feedback. *IEEE Trans. Auto. Control*, 44(5):1049–1053, May 1999.